International Atomic Energy Agency

and

United Nations Educational Scientific and Cultural Organization

INTERNATIONAL FOR THEORETICAL PHYSICS

AN INVESTIGATION OF SINGULAR LAGRANGIANS AS FIELD SYSTEMS

Eqab M. Rabei[1]
International Centre for Theoretical Physics, Trieste, Italy.

ABSTRACT

The link between the treatment of singular Lagrangians as field systems and the general approach is studied. It is shown that singular Lagrangians as field systems are always in exact agreement with the general approach.

Two examples and the singular Lagrangian with zero rank Hessian matrix are studied. The equations of motion in the field systems are equivalent to the equations which contain acceleration, and the constraints are equivalent to the equations which do not contain acceleration in the general approach treatment.

MIRAMARE – TRIESTE

July 1995

---

[1]Permanent address: Department of Physics, Mutah University, P.O. Box 7, Karak, Jordan. FAX: (962–6) 654061

1   INTRODUCTION

In previous papers [1–4] the Hamilton–Jacobi formulation of singular systems has been studied. This formulation leads us to the following total differential equations:

$$dq_a = \frac{\partial H'_\alpha}{\partial p_a}\, dx_\alpha, \quad dp_a = -\frac{\partial H'_\alpha}{\partial q_a}\, dx_\alpha, \quad dp_\alpha = -\frac{\partial H'_\beta}{\partial x_\alpha}\, dx_\beta \tag{1.1}$$

$$\alpha, \beta = 0, 1, \ldots, r; \qquad a = 1, 2, \ldots, n - r$$

with constraints

$$H'_\alpha = H_\alpha(x_\beta, q_a, p_a) + p_\alpha \tag{1.2}$$

(Note that we are adopting summation convention in this work.) Solutions of Eqs.(1.1) give the field, q_a, in terms of independent coordinates

$$q_a \equiv q_a(t, x_\mu), \qquad \mu = 1, 2, 3, \ldots, r \tag{1.3}$$

where $x_0 = t$. The link between the Hamilton–Jacobi approach and the Dirac approach is studied in Ref.[5].

In Ref.[6] the singular Lagrangians are treated as continuous systems. The Euler–Lagrange equations of constrained systems are proposed in the form

$$\frac{\partial}{\partial x_\alpha} \left[\frac{\partial L'}{\partial(\partial_\alpha q_a)} \right] - \frac{\partial L'}{\partial q_a} = 0; \quad \partial_\alpha q_a = \frac{\partial q_a}{\partial x_\alpha} \ . \tag{1.4}$$

with constraints

$$dG_0 = -\frac{\partial L'}{\partial t}\, dt \tag{1.5}$$

$$dG_\mu = -\frac{\partial L'}{\partial x_\mu}\, dt \tag{1.6}$$

where

$$L'(x_\alpha, \partial_\alpha q_a, \dot{x}_\mu, q_a) \equiv L(q_a, x_\alpha, \dot{q}_a = (\partial_\alpha q_a)\dot{x}_\alpha) \tag{1.7}$$

$$\dot{x}_\mu = \frac{dx_\mu}{dt}; \qquad \dot{x}_0 = 1$$

$$G_\alpha = H_\alpha \left(q_a, x_\beta, p_a = \frac{\partial L}{\partial \dot{q}_a} \right) \tag{1.8}$$

The variation of constraints (1.5,6) should be taken into consideration in order to have a consistent theory.

An instructive work is the canonical formalism for degenerate Lagrangians [7–9]; the starting point of this formalism is to consider Lagrangians with ranks of the Hessian

matrix less than n. Shanmugadhasan has called these systems as degenerate. For such systems some of the Euler–Lagrange equations do not contain acceleration. Following Refs.[7–9] these equations are considered as constraints. In other words, if the rank of Hessian matrix is $(n-r)$, with $r < n$, then the Euler–Lagrange equations can be expressed in the form

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_a}\right) - \frac{\partial L}{\partial q_a} = 0 \tag{1.9}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_\mu}\right) - \frac{\partial L}{\partial x_\mu} = 0 \tag{1.10}$$

With the aid of Eq.(1.9), Eq.(1.10) can be identically satisfied, i.e. $0 = 0$, or they lead to equations free from acceleration. These equations are divided into two types: type–A which contains coordinates only and type–B which contains coordinates and velocities [9]. The total time derivative of the above two types of constraints should be considered in order to have a consistent theory.

In this paper we would like to study the link between the treatment of singular Lagrangians as field systems [6] and the well–known Lagrangian formalism. In Section 2 the relation between the two approaches is discussed, and in Section 3 two examples of singular Lagrangians are constructed and solved using the two approaches. In Section 4 the treatment of a singular Lagrangian with Hessian matrix of zero rank is discussed.

2   THE RELATION BETWEEN THE TWO APPROACHES

One should notice that Eqs.(1.4) are equivalent to Eqs.(1.9). In other words the expressions

$$(\partial_\alpha q_a)\,\dot{x}_\alpha \tag{2.1}$$

and

$$\partial_\beta(\partial_\alpha q_a\,\dot{x}_\alpha)\,\dot{x}_\beta \tag{2.2}$$

In Eqs.(1.4) can be be replaced by $\dot{q}_a$ and $\ddot{q}_a$ respectively in order to obtain Eqs.(1.9). Following Refs.[1–6], we have

$$G_0 \equiv H_0\ , \tag{2.3}$$

and

$$G_\mu \equiv H_\mu = -p_\mu \tag{2.4}$$

Thus, Eqs.(1.6) lead to

$$\frac{dp_\mu}{dt} = \frac{\partial L}{\partial x_\mu} \tag{2.5}$$

3

Making use of the definition of momenta, Eqs.(2.5) lead to Eqs.(1.10). Hence Eqs.(1.5,6) are equivalent to Eqs.(1.9,10).

3 EXAMPLES

The procedure described in Section 2 will be demonstrated by the following examples.

A. Let us consider a Lagrangian of the form

$$L = \frac{1}{2} \dot{q}_1^2 + \dot{q}_1 \dot{q}_2 + \frac{1}{2} \dot{q}_2^2 + 4q_1\dot{q}_2 + (2q_1^2 + 4q_1q_2) \tag{3.1}$$

The Euler–Lagrange equations then read as

$$\ddot{q}_1 + \ddot{q}_2 - 4\dot{q}_2 - 4(q_1 + q_2) = 0 \tag{3.2}$$

$$\ddot{q}_1 + \ddot{q}_2 + 4\dot{q}_1 - 4q_1 = 0 \tag{3.3}$$

Substituting Eq.(3.2) in Eq.(3.3), gives a B–type constraint

$$B_1 = \dot{q}_2 + \dot{q}_1 + q_2 = 0 \tag{3.4}$$

For consistent theory, the time derivative of Eq.(3.4) should be equal to zero. This leads to the new B–type constraint

$$B_2 = 5\dot{q}_2 + 4(q_2 + q_1) = 0 \tag{3.5}$$

Taking the time derivative for the new constraints we get a second order differential equation for q_2

$$5\ddot{q}_2 - 4q_2 = 0 \tag{3.6}$$

which has the following solution

$$q_2 = A\, e^{\frac{2}{\sqrt{5}} t} + B\, e^{-\frac{2}{\sqrt{5}} t}$$

Now, let us look at this Lagrangian as a field system. Since the rank of the Hessian matrix is one, the above Lagrangian can be treated as a field system in the form

$$q_1 = q_1(t, q_2); \qquad x_2 = q_2 \tag{3.7}$$

Thus, the expression

$$\dot{q}_1 = \frac{\partial q_1}{\partial t} + \frac{\partial q_1}{\partial q_2}\, \dot{q}_2 \tag{3.8}$$

4

can be replaced in Eq.(3.1) to obtain the following modified Lagrangian L':

$$L' = \frac{1}{2}\left[\frac{\partial q_1}{\partial t} + \frac{\partial q_1}{\partial q_2}\,\dot{q}_2\right]^2 + \left[\frac{\partial q_1}{\partial t} + \frac{\partial q_1}{\partial q_2}\,\dot{q}_2\right]\dot{q}_2$$
$$+ \frac{1}{2}\,\dot{q}_2^2 + 4q_1\dot{q}_2 + (2q_1^2 + 4q_1q_2) \tag{3.9}$$

Making use of Eqs.(1.4), we have

$$\frac{\partial^2 q_1}{\partial t^2} + 2\frac{\partial^2 q_1}{\partial t\,\partial q_2}\,\dot{q}_2 + \frac{\partial q_1}{\partial q_2}\,\ddot{q}_2 + \frac{\partial^2 q_1}{\partial q_2^2}\,\dot{q}_2^2 + \ddot{q}_2 - 4\dot{q}_2 - 4(q_1 + q_2) = 0 \tag{3.10}$$

Note that we have made the substitution $\alpha = 0, 2$ and $a = 1$, in order to get the above equation. Making use of Eq.(3.8) and the fact that

$$\ddot{q}_1 = \frac{\partial^2 q_1}{\partial t^2} + 2\frac{\partial^2 q_1}{\partial t\,\partial q_2}\,\dot{q}_2 + \frac{\partial q_1}{\partial q_2}\,\ddot{q}_2 + \frac{\partial^2 q_1}{\partial q_2^2}\,\dot{q}_2^2 \tag{3.11}$$

Eq.(3.10) will be the same as Eq.(3.2).

According to Refs.[1–4] the quantity H_2 can be calculated as

$$H_2 = -(\dot{q}_1 + \dot{q}_2 + 4q_1) \tag{3.12}$$

Hence,

$$G_2 = -\left[\frac{\partial q_1}{\partial t} + \frac{\partial q_1}{\partial q_2}\,\dot{q}_2\right] - \dot{q}_2 - 4q_1 \tag{3.13}$$

and taking the total differential of Eq.(3.13) one gets,

$$dG_2 = -\left\{\left[\frac{\partial^2 q_1}{\partial t^2} + 2\frac{\partial^2 q_1}{\partial q_2\,\partial t}\,\dot{q}_2 + \frac{\partial^2 q_1}{\partial q_2^2}\,\dot{q}_2 + \frac{\partial q_1}{\partial q_2}\,\ddot{q}_2\right] + \right.$$
$$\left. +4\left[\frac{\partial q_1}{\partial t} + \frac{\partial q_1}{\partial q_2}\,\dot{q}_2\right] + \ddot{q}_2\right\}dt \tag{3.14}$$

Replacing the expression in the first parenthesis from Eq.(3.10) one gets

$$dG_2 = -\left\{4\dot{q}_2 + 4[q_1 + q_2] + 4\left[\frac{\partial q_1}{\partial t} + \frac{\partial q_1}{\partial q_2}\,\dot{q}_2\right]\right\}dt \tag{3.15}$$

Making use of Eq.(1.6), one finds

$$dG_2 = -4q_1\,dt \tag{3.16}$$

Equating Eq.(3.16) with Eq.(3.15), we have the following constraint

$$F_1 = \dot{q}_2 + q_2 + \frac{\partial q_1}{\partial t} + \frac{\partial q_1}{\partial q_2}\,\dot{q}_2 = 0 \tag{3.17}$$

Using the expression (3.8), one observes that this constraint is equivalent to the B–type constraint (3.4).

For a valid theory, the variation of F_1 should be zero; thus one gets

$$dF_1 = F_2 \ dt = 0 \tag{3.18}$$

where

$$F_2 = 5\dot{q}_2 + 4q_1 + 4q_2 = 0 \tag{3.19}$$

This is B_2 constraint defined in Eq.(3.5).

Again taking the total differential of the new constraint F_2, we have

$$dF_2 = [5\ddot{q}_2 - 4q_2] \ dt = 0 \tag{3.20}$$

Thus

$$5\ddot{q}_2 - 4q_2 = 0 \tag{3.21}$$

This is a second order differential equation for q_2 and is the same as Eq.(3.6). In addition, the function G_0 can be evaluated and

$$G_0 = \frac{1}{2} \left[\frac{\partial q_1}{\partial t} + \frac{\partial q_1}{\partial q_2} \ \dot{q}_2 \right]^2 + \frac{1}{2} \ \dot{q}_2^2 + \left[\frac{\partial q_1}{\partial t} + \frac{\partial q_1}{\partial q_2} \ \dot{q}_2 \right] \ \dot{q}_2 - 2q_1^2 - 4q_1 q_2 \tag{3.22}$$

where

$$dG_0 = 4\dot{q}_2 \ F_1 \ dt \tag{3.23}$$

and this does not lead to any further constraints.

B. Consider the Lagrangian of the form

$$L = \frac{1}{2} \ (\dot{q}_1^2 + \dot{q}_2^2) + \dot{q}_1 \dot{q}_2 + \frac{1}{2} \ (q_1^2 + q_2^2) \tag{3.24}$$

Then, the Euler–Lagrange equations are given as

$$\ddot{q}_1 + \ddot{q}_2 - q_1 \ = \ 0 \tag{3.25}$$

$$\ddot{q}_1 + \ddot{q}_2 - q_2 \ = \ 0 \tag{3.26}$$

Expressing Eq.(3.25) as

$$q_1 = \ddot{q}_1 + \ddot{q}_2 \tag{3.27}$$

and substituting in Eq.(3.26), one gets an A–type constraint

$$A_1 = q_1 - q_2 = 0 \tag{3.28}$$

There are no further constraints. Thus Eq.(3.26) takes the form

$$2\ddot{q}_2 - q_2 = 0 \tag{3.29}$$

6

As in the previous example, this system can be treated as field system, and the modified Lagrangian L' reads as

$$L' = \frac{1}{2}\left[\frac{\partial q_1}{\partial t} + \frac{\partial q_1}{\partial q_2}\,\dot{q}_2\right]^2 + \frac{1}{2}\,\dot{q}_2^2 + \left[\frac{\partial q_1}{\partial t} + \frac{\partial q_1}{\partial q_2}\,\dot{q}_2\right]\,\dot{q}_2 + \frac{1}{2}\,(q_1^2 + q_2^2) \tag{3.30}$$

The Euler–Lagrange equation for this field system is obtained as

$$\frac{\partial^2 q_1}{\partial t^2} + 2\frac{\partial^2 q_1}{\partial t\,\partial q_2}\,\dot{q}_2 + \frac{\partial q_1}{\partial q_2}\,\ddot{q}_2 + \frac{\partial^2 q_1}{\partial q_2^2}\,\dot{q}_2^2 + \ddot{q}_2 - q_1 = 0 \tag{3.31}$$

Again replacing $\ddot{q}_1$ by the expression (3.11), Eq.(3.31) will be the same as Eq.(3.25). Besides, the function G_2 can be calculated as

$$G_2 = -\left[\frac{\partial q_1}{\partial t} + \frac{\partial q_1}{\partial q_2}\,\dot{q}_2\right] - \dot{q}_2 \tag{3.32}$$

and the total differential of G_2 can be written as

$$dG_2 = -\left[\frac{\partial^2 q_1}{\partial t^2} + 2\frac{\partial^2 q_1}{\partial t\,\partial q_2}\,\dot{q}_2 + \frac{\partial^2 q_1}{\partial \dot{q}_2^2}\,\dot{q}_2^2 + \frac{\partial q_1}{\partial q_2}\,\ddot{q}_2 + \ddot{q}_2\right]\,dt \tag{3.33}$$

Using Eq.(3.31) and Eq.(1.6), we have

$$dG_2 = -q_1\;dt = -q_2\;dt \tag{3.34}$$

which leads to the following constraint

$$F_1 = q_1 - q_2 = 0 \tag{3.35}$$

This is an A–type constraint of the form (3.28).

Taking the total differential of F_1, we have

$$dF_1 = \left[\frac{\partial q_1}{\partial t} + \frac{\partial q_1}{\partial q_2}\,\dot{q}_2 - \dot{q}_2\right]\,dt = 0 \tag{3.36}$$

and this leads to a new constraint

$$F_2 = \frac{\partial q_1}{\partial t} + \frac{\partial q_1}{\partial q_2}\,\dot{q}_2 - \dot{q}_2 = 0 \tag{3.37}$$

which is equivalent to the total time derivative of the constraint (3.28).

Again calculating the total differential of F_2, one gets

$$dF_2 = \left[\frac{\partial^2 q_1}{\partial t^2} + 2\frac{\partial^2 q_1}{\partial t\,\partial q_2}\,\dot{q}_2 + \frac{\partial^2 q_1}{\partial q_2^2}\,\dot{q}_2^2 + \frac{\partial q_1}{\partial q_2}\,\ddot{q}_2 - \ddot{q}_2\right]\,dt = 0 \tag{3.38}$$

and making use of Eqs.(3.31) and (3.35), we get

$$2\ddot{q}_2 - q_2 = 0 \tag{3.39}$$

which is the same as Eq.(3.29). Besides the function G_0 is calculated as

$$G_0 = \frac{1}{2}\left[\frac{\partial q_1}{\partial t} + \frac{\partial q_1}{\partial q_2}\,\dot{q}_2\right]^2 + \frac{1}{2}\,\dot{q}_2^2 + \left[\frac{\partial q_1}{\partial t} + \frac{\partial q_1}{\partial q_2}\,\dot{q}_2\right]\,\dot{q}_2 - \frac{1}{2}\,[q_1^2 + q_2^2] \tag{3.40}$$

Thus, the total differential of G_0 is obtained as

$$dG_0 = \dot{q}_2\,F_1\,dt = 0 \tag{3.41}$$

and with the aid of Eq.(3.35), it is identically satisfied.

4 A SINGULAR LAGRANGIAN WITH ZERO RANK HESSIAN MATRIX

According to the treatment of singular Lagrangians as field systems: if the Hessian matrix has rank equal to zero, the Lagrangian cannot be treated as field system. Whereas, the equation of motion which does not contain acceleration can be obtained using the constraints (1.6).

As an example let us consider the following Lagrangian which was given in Ref.[10]:

$$L = (q_2 + q_3)\,\dot{q}_1 + q_4\,\dot{q}_3 + \frac{1}{2}\,(q_4^2 - 2q_2 q_3 - q_3^2) \tag{4.1}$$

The momenta are obtained as

$$p_1 = q_2 + q_3, \quad p_2 = 0, \quad p_3 = q_4, \quad p_4 = 0 \tag{4.2}$$

Thus,

$$G_1 = -(q_2 + q_3) \tag{4.3}$$
$$G_2 = 0 \tag{4.4}$$
$$G_3 = -q_4 \tag{4.5}$$
$$G_4 = 0 \tag{4.6}$$

Making use of Eqs.(1.6), one gets

$$dG_1 = -(\dot{q}_2 + \dot{q}_3)\,dt = 0 \tag{4.7}$$
$$dG_2 = 0 = -(\dot{q}_1 - \dot{q}_3)\,dt \tag{4.8}$$
$$dG_3 = -\dot{q}_4\,dt = -(\dot{q}_1 - q_2 - q_3)\,dt \tag{4.9}$$
$$dG_4 = 0 = -(\dot{q}_3 + q_4)\,dt \tag{4.10}$$

8

These equations lead to the following equations of motion

$$\dot{q}_2 + \dot{q}_3 = 0 \tag{4.11}$$

$$\dot{q}_1 - q_3 = 0 \tag{4.12}$$

$$\dot{q}_4 - \dot{q}_1 + q_2 + q_3 = 0 \tag{4.13}$$

$$\dot{q}_3 + q_4 = 0 \tag{4.14}$$

and these are the Euler–Lagrange equations which are free from acceleration, and are of B–type constraints.

5   CONCLUSIONS

As it was mentioned in the introduction if the rank of the Hessian matrix for discrete systems is $(n - r); 0 < r < n$, the systems can be treated as field systems. It can be observed that the treatment of Lagrangians as field systems is always in exact agreement with the general approach. The equations of motion (1.4) are equivalent to the equations of motion (1.9). Besides, the constraints (1.6) are equivalent to the equations (1.10).

The consistent theory in the treatment of Lagrangians as field systems also leads to two types of constraints: a B–type which contains at least one member of the set $\left\{\dot{q}_\mu, \frac{\partial q_a}{\partial t}, \frac{\partial q_a}{\partial q_\mu}\right\}$, and an A–type which contains coordinates only. As we have seen, in the first example F_1 and F_2 are B–types; while the constraint F_1 in the second example is an A–type.

In the general approach the constraints can be obtained from the Euler–Lagrange equations, whereas, in the treatment of Lagrangians as field systems, the constraints can be determined from the relations (1.5,6) and the new constraints can be obtained using the variations of these relations.

References

[1] Y. Güler, Nuovo Cimento B **107**, 1389 (1992).

[2] Y. Güler, Nuovo Cimento B **107**, 1143 (1992).

[3] Eqab M. Rabei and Y. Güler, Phys. Rev. A **46**, 3513 (1992).

[4] Eqab M. Rabei and Y. Güler, Turkish J. Phys. **16**, 297 (1992).

[5] Eqab M. Rabei, On Hamiltonian Systems with constraints, to appear in Hadronic Journal, Vol.18 (1995).

[6] N.I. Farahat and Y. Güler, Phys. Rev. A **51**, 68 (1995).

[7] S. Shanmugadhasan, Proc. Camb. Phil. Soc. **59**, 743 (1963).

[8] S. Shanmugadhasan, J. Math. Phys. **14**, 677 (1973).

[9] K. Sundermeyer, Lecture Notes in Physics, Springer–Verlag, (1982).

[10] Usha Kulshreshtha, J. Math. Phys. **33**, 633 (1992).